Fractal dimension and degree of order in sequential deposition of mixture

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We present a number of models describing the sequential deposition of a mixture of particles whose size distribution is determined by the power law $p(x) \sim \alpha x^{\alpha-1}$, $x \leqslant l$. We explicitly obtain the scaling function in the case of random sequential adsorption and show that the pattern created in the long-time limit becomes scale invariant. This pattern can be described by a unique exponent, the fractal dimension. In addition, we introduce an external tuning parameter β to describe the correlated sequential deposition of a mixture of particles where the degree of correlation is determined by β, while $\beta=0$ corresponds to the random sequential deposition of a mixture. We show that the fractal dimension of the resulting pattern increases as β increases and reaches a constant nonzero value in the limit $\beta \to \infty$ when the pattern becomes perfectly ordered or nonrandom fractals. [S1063-651X(97)09905-4]

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I. INTRODUCTION

The formation of stochastic fractals is an active field of research both theoretically and experimentally. Yet the mechanism by which nature creates fractals and the relationship between the degree of order and the fractal dimension is poorly understood. The history of describing natural objects by geometry is as old as the science itself. However, traditionally Euclidean lines, squares, rectangles, circles, spheres, etc., have been the basis of our intuitive understanding of the geometry of almost all natural objects. But nature is not restricted to Euclidean space. Instead, most of the natural objects we see around us are so complex in shape that conventional Euclidean space is not sufficient to describe them. It appears to be essential to invoke the concept of fractal geometry to characterize such complex objects quantitatively. It further enables us to search for symmetry and order even in disordered, complex systems [1,2]. The importance of the discovery of fractals can hardly be exaggerated. Yet there is no neat and complete definition of a fractal. Instead one associates a fractal with a shape made of parts similar to the whole in some way. It is typically quantified by a noninteger exponent called the fractal dimension that can uniquely characterize the structure. This definition immediately confirms the existence of scale invariance, that is, objects look the same on different scales of observation. To understand fractals, their physical origin, and how they appear in nature we need to be able to model them theoretically. This forms part of our motivation of the present work.

The simplest way to construct a fractal is to deterministically repeat a given operation. The construction of a classical Cantor set is a simple textbook example of such a fractal. It is created by fragmenting a line into n equal pieces and removing $n-m$ of the parts created and repeating the process with the m remaining pieces [1]. This process is repeated *ad infinitum*. However, this construction is too artificial as it differs in two ways from the fractals that occur in nature. It does not have any kinetics and it does not have any randomness.

In this work we introduce a stochastic process allowing a number of intrinsic tuning parameters that may be considered as a natural kinetic counterpart of the classical Cantor construction and is a potential candidate in order to understand the essential governing rule of creating complex objects. These intrinsic tuning parameters are used to determine the degree of randomness and the rate at which a given operation is repeated to create a fractal. The interval is chosen for breaking stochastically, and once an interval is chosen the cuts are placed randomly on the interval, while the degree of randomness is determined by the precise choice of deposition kernels. Thus, starting with an infinitely long interval, what remains in the long-time limit is an infinite number of points scattered over the intervals. The properties of these points create a set that appears to be statistically self-similar and is characterized by a fractal dimension.

The construction of stochastic fractals we consider is not at all pedagogical. One immediate and potential application of the stochastic fractal is the sequential deposition of a mixture of particles with a continuous distribution of sizes. However, the model we consider in this work mimics the configuration when objects once inserted are clamped in their spatial positions for which nonequilibrium configurations are generated. The random sequential absorption processes have been found to describe many experimental systems, namely, the adhesion of proteins and colloidal particles to uniform surfaces [3,4], the reaction of various polymer chain systems such as methyl vinyl ketone [5], and many fields in chemistry and physics. Although the process is conceptually simple, understanding its kinetics analytically is a challenging problem (see an excellent review article [6]). The deposition of particles of definite sizes in one dimension has been solved exactly and analytically in both continuous and discrete cases. The continuous version of this model is known as the random car parking problem. Recently, the deposition of a mixture of a small number of definite-sized particles has been considered [7–9]. In Ref. [10] Bartelt and Privman solved exactly the one-dimensional lattice version for depo-

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sition of a mixture when it is composed of fixed-size and pointlike particles. These studies reveal that if the mixture contains a small number of different sizes then the various geometric kinetic characteristics are primarily determined by the smallest size. From the experimental point of view, the deposition kinetics of well-defined mixtures of different types of particles is of great importance. It is well known that if objects are of definite size and once deposited are clamped in their positions, nonequilibrium configurations are created with a strong nonergodic and non-Markovian nature [11]. In this case, the resulting system does not produce a scale-invariant pattern. Instead, it reaches a jamming limit when it is impossible to place further objects without overlapping. Hence a unique number (the jamming limit or the coverage) is sufficient to characterize the resulting pattern created in the long-time limit. If a mixture contains a continuous distribution of sizes and is deposited sequentially such that once deposited it is clamped to its position, then it is equivalent to our stochastic kinetic Cantor construction [12]. Hence the deposition of a mixture of particles will create a scale-invariant pattern that does not reach the jamming limit since there always exists a particle if there is an uncovered space during the process.

Interesting questions arise from the present work. (i) What is the role of the fractal dimension during pattern formation? (ii) Is there any relation between fractal dimension and degree of order? (iii) What are the relevant parameters to tune the degree of order and what are their physical meaning? The present work is an attempt to answer these questions. In Sec. II we present the general equation to describe sequential deposition of an arbitrary number of particles. In Sec. III we present model I describing random sequential deposition of a mixture of different-sized particles at a rate determined by the kernel $F(x,y|z)=(x+y+z)^{\gamma}$, where the exponent γ determines only the deposition rate and does not play any role in determing the degree of randomness. The kernel $F(x,y|z)$ is defined as the rate with which an interval $(x+y+z)$ is destroyed by depositing a particle of size z, creating two new intervals x and y. The parking distribution function $p(z)$ is the probability of attempting to deposit a particle of size z. The deposition of a mixture of different-sized particles is achieved by choosing a $p(z)$ power-law form of parking distribution function with respect to z. We obtain an explicit scaling solution and show that the resulting pattern in the long-time limit is scale invariant in both time and space. In Sec. IV we consider correlated sequential deposition of a mixture of particles (model II) by choosing $F(x,y|z)=xy$. This parabolic choice implies that particles are more likely to deposit on the middle of an interval. We further obtain the fractal dimension of the resulting pattern for model II and compare it with the fractal dimension for model I. Finally, in Sec. V we present a generalized version of both models to describe correlated sequential deposition of an arbitrary number of particles where the degree of correlation is controlled by a parameter β and obtain the fractal dimension as a function of β. In Sec. VI we summarize the results and discuss various points in order to reach a conclusion.

II. SEQUENTIAL DEPOSITION OF AN ARBITRARY NUMBER OF PARTICLES

The connection between the one-dimensional model of car parking and the fragmentation processes was emphasized by Ziff [13]. The fragmentation process can be thought of as the deposition of points on a line whose position depends on the kinetic rule defined by the choice of kernels. Let $\psi(x,t)$ be the concentration of empty or uncovered intervals of length x at time t. Then the rate equation for this concentration obeys the integro-differential equation

$$\frac{\partial\psi(x,t)}{\partial t}=-\psi(x,t)\int\prod_{j=1}^{n-m}p(x_j)F(x_{n-m+1},\ldots,x_n|x_1,\ldots,x_{n-m})\,\delta\left(x-\sum_{i=1}^{n}x_i\right)\times\prod_{i=1}^{n}dx_i+m\int\psi(y,t)F(x,x_{n-m+1},\ldots,x_{n-1}|x_1,\ldots,x_{n-m})\,\delta\left(y-x-\sum_{i=1}^{n-1}x_i\right)\prod_{i=1}^{n-1}dx_i dy\prod_{j=1}^{n-m}p(x_j). \quad (1)$$

Here $n=3,4,\ldots,$ and $m=2,3,\ldots,$ so that $n-m=1,2,3,\ldots$ is the number of particles that are deposited at each time step. Notice that the m value in particular puts a strong constraint on the depositing particles. That is, if $n-m=i$ and $m<i+1$, then the precise m will determine how many of the depositing particles must deposit next to each other so that they produce only m new empty spaces. The term $p(x_i)$ determines the size of the depositing particles at each time step and $F(x_{n-m+1},\ldots,x_n|x_1,\ldots,x_{n-m})$ determines the rate and the rules with which $x_1,\ldots,x_{n-m}$ particles are to be deposited to create m new empty spaces at each time step. The first term on the right-hand side of Eq. (1) represents the destruction of spaces of size x and the second term represents their creation from bigger spaces. Equation (1) can also describe the process of breaking an interval into n pieces and throwing away $n-m$ pieces to create a stochastic fractal.

III. MODEL I

We first choose to consider the deposition of one particle at each time step $(m=2)$. The rate equation (1) then becomes

$$\frac{\partial\psi(x,t)}{\partial t}=-\psi(x,t)\int_0^x p(z)dz\int_0^{x-z}dyF(x-y-z,y|z)+2\int_x^{\infty}dy\,\psi(y,t)\int_0^{y-x}dz\,p(z)F(x,y-x-z|z). \quad (2)$$

We choose the size of the depositing particles following the power-law form

$$p(x)=\begin{cases}\alpha x^{\alpha-1}, & x\leqslant l\\ 0, & x>l,\end{cases} \tag{3}$$

where $\alpha>0$ in order to ensure that $p(z)$ is normalized. This form of the parking distribution implies the deposition of a mixture of particles with a continuous distribution of sizes. We further choose the rate with which particles are deposited to be

$$F(x_2,x-x_1-x_2|x_1)=x^{\gamma}. \tag{4}$$

This choice of kernel implies that any point in the empty spaces is equally likely to be chosen for deposition by a particle whose size is determined by the choice of $p(x)$. However, the empty spaces are picked at each time step where the particles are to be deposited, which is determined by the exponent γ. The rate equation then becomes

$$\frac{\partial\psi(x,t)}{\partial t}=-\frac{x^{\alpha+\gamma+1}}{\alpha+1}\psi(x,t)+2\int_x^l\psi(y,t)y^{\gamma}(y-x)^{\alpha}dy \quad \text{for } x\leqslant l. \tag{5}$$

Notice that setting $\alpha=0$ or choosing $p(z)=\delta(z)$ describes the random deposition of zero-sized particles on a line at a rate $x^{\gamma+1}$, i.e., Eq. (5) becomes the standard binary fragmentation process [14]. We further notice that we can let l be infinitely long without loss of generality, in which case the upper limit of the second term in Eq. (5) must be infinity. It is well known that in one dimension the random parking of cars of definite length is highly nonergodic in the sense that the whole space is not visited by the depositing particles. However, in the case of the random deposition of a mixture of particles with a power-law form of distribution sizes, the system retains an ergodic nature. In fact, the size of the particles to be deposited depends intrinsically on the size of the available empty spaces. Consequently, the size of the deposited particles becomes pointlike in the long-time limit and there is always an available particle to deposit whatever the size of the empty space in the system. The second term on the right-hand side of Eq. (5) reveals that as α increases the nonlocal character becomes more and more prominent, so that it becomes increasingly difficult to solve the equation. Notice that the exponent γ does not play any role in determining the degree of nonlocality. A closer look at Eq. (5) reveals that for $\alpha+\gamma+1<0$ the deposition processes can be fast enough such that subsequent generation of intervals has a shorter lifetime than the previous generation. In other words, smaller intervels are more likely to be picked for deposition than larger intervals, which is not physically interesting. Therefore, we restrict the deposition process to only $\alpha+\gamma+1>0$.

We define the moment of the empty size distribution function as

$$M(q,t)=\int_0^l\psi(x,t)x^q dx. \tag{6}$$

We now multiply both sides of the rate equation by x^q and integrate over x, thus obtaining a rate equation for the moments

$$\frac{dM(q,t)}{dt}=\left(2\frac{\Gamma(q+1)\Gamma(\alpha+1)}{\Gamma(q+\alpha+2)}-\frac{1}{\alpha+1}\right)\times M(q+\alpha+\gamma+1,t). \tag{7}$$

This equation can be solved to find the solution for the qth moment, the solution taking the form of a generalized hypergeometric function with the numerator $\alpha+1$ and denominator parameters. In order to understand the various physical aspects of the problem within the simplest possible way, we first consider the case when $\alpha=1$. In this case the rate equation becomes

$$\frac{\partial\psi(x,t)}{\partial t}=-\frac{x^{\gamma+2}}{2}\psi(x,t)+2\int_x^l\psi(y,t)y^{\gamma}(y-x)dy. \tag{8}$$

This class of models includes the one that is considered in [12,16] when $\gamma=0$. Iterating the above equation to get all the higher derivatives and substituting in the Taylor-series expansion of $M(q,t)$ about $t=0$ yields

$$M(q,t)=l^q{}_2F_2\left(\frac{q-a}{\gamma+2},\frac{q+a+3}{\gamma+2};\frac{q+1}{\gamma+2},\frac{q+2}{\gamma+2};-tl^{\gamma+2}\right), \tag{9}$$

where ${}_2F_2(a,b;c,d;x)$ is the generalized hypergeometric function [17,18] and $a=(-3+\sqrt{17})/2=0.561\,528\,8$. The asymptotic expansion of the generalized hypergeometric function immediately reveals that the moments show the power-law behavior

$$M(q,t)\sim t^{(q-a)/(\gamma+2)}. \tag{10}$$

Notice that the exponent of the asymptotic expression for the moment is linear in q, which reveals the existence of simple scaling.

A. Explicit scaling solution

In this section we attempt to show that in the long-time limit the empty size distribution shows a power-law behavior. A linear power-law behavior of the moments reveals that the system reaches a scaling or self-similar behavior does not depend on the initial conditions for which one can invoke universality. However, a closer look at the rate equation further reveals that only one of the two parameters has independent dimension, i.e., x and t^z have the same dimension where the exponent z is yet to be determined. That is, the dimension of ψ must be expressible in terms of the independent parameter t or x alone. We can therefore define the dimensionless quantities as

$$\xi=xt^{-1/z} \quad \text{or} \quad \zeta=tx^z \tag{11}$$

and

$$\phi(xt^{1/z})=\frac{\psi(x,t)}{t^{\nu/z}},\quad \Phi(tx^z)=\frac{\psi(x,t)}{x^{-\theta}}. \tag{12}$$

If scaling theory is obeyed, a plot of $\psi(x,t)/t^{\nu/z}$ against ξ or a plot of $\psi(x,t)/x^{-\theta}$ against ζ should fall on a single curve for any initial distribution. This reveals also that a self-similar solution in time and space exists. However, we find it convenient to consider the scaling ansatz

$$\psi(x,t)=x^{-\theta}\Phi(tx^z) \tag{13}$$

for which the pattern that develops in the long-time limit is self-similar in space. The exponent θ can be obtained from the rate equation for the moments using the condition for moment that is time independent to give $\theta=1+a$. We now substitute this ansatz into the rate equation to obtain

$$t^{(\gamma+2-z)/z}=\frac{-\frac{1}{2}\zeta^{(\gamma+2-\theta)/z}\Phi(\zeta)-\frac{2}{z}\zeta^{1/z}\int_\zeta^\infty\eta^{(\gamma+1-z-\theta)/z}\Phi(\eta)d\eta+\frac{2}{z}\int_\zeta^\infty\eta^{(\gamma+2-z-\theta)/z}\Phi(\eta)d\eta}{\zeta^{(z-2)/z}\Phi'(\zeta)}. \tag{14}$$

Demanding the scaling solution to exist, we find that $z=\gamma+2$. The equation that we need to solve to find the scaling solution is

$$\zeta^{(\gamma+2-\theta)/(\gamma+2)}\Phi'(\zeta)+\frac{1}{2}\zeta^{(\gamma+2-\theta)/(\gamma+2)}\Phi(\zeta)$$
$$=\frac{2}{\gamma+2}\int_\zeta^\infty\eta^{-\theta/(\gamma+2)}\Phi(\eta)d\eta$$
$$-\frac{2}{\gamma+2}\zeta^{1/(\gamma+2)}\int_\zeta^\infty\eta^{-(\theta+1)/(\gamma+2)}\Phi(\eta)d\eta. \tag{15}$$

In order to eliminate the integral we differentiate this equation twice with respect to ζ to reduce it to the third-order differential equation

$$\zeta^2\Phi'''(\zeta)+\zeta\left[\left(3-\frac{2\theta+1}{\gamma+2}\right)+\frac{\zeta}{2}\right]\Phi''(\zeta)+\left[\left(1-\frac{\theta}{\gamma+2}\right)\right.$$
$$\left.\times\left(1-\frac{\theta+1}{\gamma+2}\right)+\frac{3\gamma+5-2\theta}{2(\gamma+2)}\zeta\right]\Phi'(\zeta)+\left[\frac{1}{2}\left(1-\frac{\theta}{\gamma+2}\right)\right.$$
$$\left.\times\left(1-\frac{\theta+1}{\gamma+2}\right)-\frac{2}{(\gamma+2)^2}\right]\Phi(\zeta)=0. \tag{16}$$

We can rescale the equation to obtain

$$\frac{\zeta^2}{4}\Phi'''(\zeta)+\frac{\zeta}{2}\left[\left(3-\frac{2\theta+1}{\gamma+2}\right)+\frac{\zeta}{2}\right]\Phi''(\zeta)+\left[\left(1-\frac{\theta}{\gamma+2}\right)\right.$$
$$\left.\times\left(1-\frac{\theta+1}{\gamma+2}\right)+\frac{3\gamma+5-2\theta}{2(\gamma+2)}\frac{\zeta}{2}\right]\Phi'(\zeta)+\left[\left(1-\frac{\theta}{\gamma+2}\right)\right.$$
$$\left.\times\left(1-\frac{\theta+1}{\gamma+2}\right)-\frac{4}{(\gamma+2)^2}\right]\Phi(\zeta)=0. \tag{17}$$

The solution of Eq. (17) is given by the generalized hypergeometric function

$$\Phi(\zeta)={}_2F_2\left(1+\frac{\sqrt{17}-2\theta-1}{2(\gamma+2)},1-\frac{\sqrt{17}+2\theta+1}{2(\gamma+2)};1\right.$$
$$\left.-\frac{\theta}{\gamma+2},1-\frac{\theta+1}{\gamma+2};-\frac{\zeta}{2}\right). \tag{18}$$

This is the explicit and exact scaling solution from which one can find the large-ζ behavior,

$$\Phi(\zeta)\sim e^{-\zeta/2}. \tag{19}$$

The two scaling functions are related through

$$\phi(xt^{1/(\gamma+2)})=(xt^{1/(\gamma+2)})^{-\theta}\Phi((xt^{1/(\gamma+2)})^{\gamma+2}). \tag{20}$$

Hence we obtain the scaling function for large $\zeta=\xi^{\gamma+2}$ as

$$\phi(\xi)\sim\xi^{-(1+a)}e^{-\xi^{\gamma+2}/2}. \tag{21}$$

One can recover the solution obtained in [16] using an indirect and different method from this general solution by setting $\gamma=0$. It is possible to obtain the scaling solution for a higher value of α, but as α increases the numerator and denominator parameter of the generalized hypergeometric function become increasingly complicated. However, the knowledge of z and θ and a detailed survey reveals that it is possible to write the scaling solution in the large-ξ limit for general α as

$$\phi(\xi)\sim\xi^{-[1+a(\alpha)]}e^{-\xi^{\alpha+\gamma+1}/2}, \tag{22}$$

where $a(\alpha)$ is the solution of the following equation for q:

$$2\frac{\Gamma(q+1)\Gamma(\alpha+1)}{\Gamma(q+\alpha+2)}=\frac{1}{\alpha+1}. \tag{23}$$

We now write the empty size distribution for the long time limit as

$$\psi(x,t)\sim x^{-[1+a(\alpha)]}\Phi(\zeta). \tag{24}$$

That is, we can choose scales $\psi_0(x)=x^{-[1+a(\alpha)]}$ depending on the spatial variable for the empty size distribution function and $t_0(x)=x^{-(\alpha+\gamma+1)}$ for the temporal variable. Therefore, in the new scale the properties of the empty size distribution function can be expressed in terms of one variable, i.e.,

TABLE I. Fractal dimension D_f of the stochastic fractals for model I; the corresponding dimensionality for the Cantor set is given in the parentheses.

	n		
m	3	4	5
2	0.5615288(0.6309)	0.4348(1/2)	0.3723(0.4307)
3	1	0.7478(0.7925)	0.6295(0.6826)
4		1	0.8315(0.8614)
5			1

$$\psi(x) \sim x^{-[1+a(\alpha)]} e^{t/t_0}. \tag{25}$$

This implies that ψ/ψ_0 and t/t_0 are self-similar coordinates.

B. Statistically self-similar pattern formation

The existence of scaling shows that the pattern created in the long-time limit becomes scale-free, i.e., the whole can be obtained from the parts by a suitable change in scale. Essentially, this implies that we can invoke the idea of a fractal dimension: a dimension that uniquely determines the geometry of the object. We now use the usual box counting method to determine the fractal dimension. We define the size of the segments to be

$$\delta = \frac{M(1,t)}{M(0,t)} \sim t^{-1/(\gamma+2)} \tag{26}$$

and we count the number of such segments needed to cover the whole set of points to determine the fractal dimension. In the limit $\delta \to 0$, we find that the number of segments $\langle N(\delta)\rangle$ required to cover the set created by Eq. (5) scales as

$$\langle N(\delta)\rangle \sim \delta^{-D_f(\alpha)}, \tag{27}$$

where $D_f(\alpha)$ is the real and positive root of the polynomial equation in q obtained from Eq. (7). Consequently, D_f is found to obey

$$2\,\frac{\Gamma(D_f(\alpha)+1)\Gamma(\alpha+1)}{\Gamma(D_f(\alpha)+\alpha+2)} = \frac{1}{\alpha+1} \tag{28}$$

Thus a single scaling exponent $D_f(\alpha)$ completely characterizes the structures of the objects, which is reminiscent of the jamming limit. Note that fractal dimension $D_f(\alpha)$ does not depend on the exponent γ, so it is independent of the rate at which particles are deposited, provided $\gamma > -(\alpha+1)$. In Table I we give a spectrum of fractal dimensions for different values of α for model I. This shows that as α increases the fractal dimension decreases. Later, we also attempt to give a physical interpretation of the exponent α.

IV. MODEL II

We shall now consider another model that follows the same parking distribution with a trivial shift of one in the exponent α, i.e., $p(z) \sim (\alpha+1)z^{\alpha}$, but different deposition rate. The deposition rate of this model is

$$F(x,y|z) = xy. \tag{29}$$

This particular choice of the deposition rate implies that all the points along the chosen empty space are not equally likely to be deposited, although all the empty spaces compete on equal footing to be chosen where particles can be deposited. That is, the rate depends on the size of the deposited particles as well as on the size of the two smaller empty spaces created due to deposition. Substituting this into Eq. (2) with $n=3$ and $m=2$, we obtain the rate equation

$$\frac{\partial\psi(x,t)}{\partial t} = -\frac{x^{\alpha+4}}{(\alpha+3)(\alpha+4)}\psi(x,t) + 2\int_x^l x(y-x)^{\alpha+2}\psi(y,t)dy \quad \text{for } x \le l. \tag{30}$$

Notice that $\alpha=-1$ describes the fragmentation process with the fragmentation kernel $F(x,y)=xy$ [19]. Also notice that the intensity of the nonlocal character is higher than the previous model, for which we find it increasingly difficult to find the scaling solution. Nevertheless, for our purpose it is enough to know that scaling exists. Substituting the definition of the moment into the above equation yields

$$\frac{dM(n,t)}{dt} = \left[\frac{\Gamma(n+2)\Gamma(\alpha+5)}{\Gamma(n+\alpha+5)} - 1\right] M(n+\alpha+4,t). \tag{31}$$

We find that the asymptotic behavior of the moment can provide some of the interesting features of the system. Hence, from now on we are only interested in finding the fractal dimension of the system that can uniquely characterize the structure. Following a similar procedure as we have done before, we find that the number of segments needed to cover the whole set created scales as

$$\langle N(\delta)\rangle \sim \delta^{-D_f(\alpha)}, \tag{32}$$

where $D_f(\alpha)$, as before, can be obtained to satisfy

$$\frac{\Gamma(D_f(\alpha)+2)\Gamma(\alpha+5)}{\Gamma(D_f(\alpha)+\alpha+5)} = 1. \tag{33}$$

Here $D_f(\alpha)$ is the fractal dimension of the pattern formed by this model. It is clear that the order of the polynomial equation is determined by the α value. Therefore, the fractal dimension $D_f(\alpha)$ is different for different α values. Comparing this model with the previous model, we find that the fractal dimension for this model is always higher than that for the previous model for each corresponding α value. Also, in both cases fractal dimensions appear to decrease monotonically as α increases. This is a feature that we shall discuss in more detail later. We intend to determine if there exists any relation between the degree of order in the pattern and the fractal dimension. In order to do this we consider a further generalization of the two models we discussed.

V. MODEL III

We now turn to the more general model in which, at each time step, more than one particle will attempt to be deposited. We choose the parking distribution and deposition rates that are of the same functional form, i.e., we choose the particle size distribution for deposition to take the form

$$p(x_i)=g(x_i) \quad \text{for } x\leqslant l, \tag{34}$$

and the deposition rate

$$F(x_{n-m+1},\ldots,x_n|x_1,\ldots,x_{n-m})=\prod_{i=n-m+1}^{n} g(x_i). \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (1), we obtain the rate equation

$$\frac{\partial\psi(x,t)}{\partial t}=-F_n(x)\psi(x,t)+mg(x)\int_x^l\psi(y,t)\times F_{n-1}(y-x)dy \quad \text{for } x\leqslant l, \tag{36}$$

where the functions $\{F_n(x),n=2,3,\ldots\}$ are defined by

$$F_n(x)=\int\delta\left(x-\sum_{i=1}^{n}x_i\right)\prod g(x_i)dx_i. \tag{37}$$

Equation (36) is equivalent to the dynamic system of breaking an interval into n pieces and throwing away $n-m$ of them at each time step [20]. It is straightforward to show that

$$F_n(x)=\int_0^x g(y)F_{n-1}(x-y)dy \tag{38}$$

for $n\geqslant 3$ and

$$F_2(x)=\int_0^x g(y)(x-y)d. \tag{39}$$

We further specify our model by choosing $g(y)=y^\beta$, with β treated as an external tunable parameter. When $g(y)=y^\beta$ we can obtain the function $\{F_n(x),n=2,3,\ldots\}$ from Eqs. (38) and (39) as

$$F_n(x)=\frac{[\Gamma(\beta+1)]^n}{\Gamma(n(\beta+1))}x^{n(\beta+1)-1}. \tag{40}$$

It is interesting to note that if we set $\beta=0$, $m=2$, and $n=\alpha+2$ in Eq. (36), we get the same rate equation for the empty size distribution as described by model I [Eq. (5)] with $\gamma=0$. It reveals that we can give an alternative interpretation of model I. That is, at each time step an interval is broken into $\alpha+2$ random pieces and α of them are removed from the system. Alternatively, we can say that the exponent α determines the number of particles to be deposited at each time step on an interval. However, for $\alpha>1$ the m value put a strong constraint on the depositing particles. That is, the process describes sequential deposition of α particles consecutively as if they were a single particle. Similarly, if we set $\beta=1$, $m=2$, and $n=(\alpha+5)/2$, we get the same rate equation as described by model II. This reveals that at each time step $(\alpha+1)/2$ are deposited consecutively. These two features help us to understand the physical role played by the parking distribution exponent α. That is, as α increases the length of the depositing particles on the average increases with respect to that of the corresponding lower α value. For further support we concentrate on the fractal dimension of the resulting pattern.

Since the moments of the empty space distribution function can characterize the fragmenting systems more easily than the empty space distribution function itself, we now consider the behavior of the moment only. For $g(y)=y^\beta$, the time evolution of the moments can be obtained using Eq. (6),

$$\frac{dM(q,t)}{dt}=[\Gamma(\beta+1)]^{n-1}\left(\frac{m\Gamma(q+\beta+1)}{\Gamma(q+n(\beta+1))}-\frac{\Gamma(\beta+1)}{\Gamma(n(\beta+1))}\right)M(q+n(\beta+1)-1,t). \tag{41}$$

This equation can be solved for the moments $M(q,t)$, the solution again taking the form of a generalized hypergeometric function with numerator $(n-1)(\beta+1)$ and denominator parameters. We now consider the scaling behavior of these models that define the scaling exponent θ and z and give the long-time dependence of the moment as $M(q,t)\sim t^{z(\theta-q-1)}$. We can immediately find z for all m, n, and α because in the long-time limit the moments behave as

$$M(q,t)\sim A(q)t^{-b(q)}. \tag{42}$$

Substituting this into the rate equation for the moment yields a difference equation

$$b(q+n(\beta+1)-1)=b(q)+1. \tag{43}$$

Iterating this and using the q value for which the moment becomes time independent, we find

$$b(q)=\frac{q-q^*}{n(\beta+1)-1}. \tag{44}$$

This gives

$$z=\frac{1}{n(\beta+1)-1} \tag{45}$$

and $q(\beta,m,n)$ related to θ by $\theta=D_f+1$ and can be obtained from Eq. (41) and satisfies

$$\frac{m\Gamma(q+\beta+1)}{\Gamma(q+n(\beta+1))}=\frac{\Gamma(\beta+1)}{\Gamma(n(\beta+1))}. \tag{46}$$

In particular, this model can be described as a correlated sequential deposition of particles on a substrate where the degree of correlation is determined by the exponent β.

For the the power-law form of the the parking distribution function this model describes the deposition of $n-m=1,2,3,\ldots$ particles and creates $m=2,3,\ldots$ new empty spaces, respectively. If $n-m=p$ and $m<p+1$ then the m value determines how many of them deposit consecutively. This deposition phenomenon can equivalently be interpreted as cutting an interval into n pieces and removing $n-m$ of the parts created and repeating the process with the remaining m pieces thus resembling the concept of classic

Cantor set. The fact that the size and position of the particles to be removed are chosen stochastically, unlike in the Cantor set where there are no kinetics, is irrelevant because we are considering the scaling regime $t\to\infty$. Since this process is repeated *ad infinitum*, it forms stochastic fractals with dimension $0\leqslant D_f\leqslant 1$. As before, we use the box counting method by defining a characteristic length δ so that we can count the number of segments required to cover the set when $\delta\to 0$, which determines the properties of the resulting set and scales as

$$\langle N(\delta)\rangle \sim \delta^{-D_f}. \tag{47}$$

Thus we find that the Hausdorff-Besicovitch dimension in this case is equal to q^*. We still have z given by Eq. (45), but now the value of q^* (and hence θ and D_f) is nontrivial. This class of fractals includes that considered in [12], with $n=3$, $m=2$, and $\beta=0$. In the limit $\beta\to\infty$ we can use Stirling's formula and the asymptotic formula

$$\Gamma(az+b)\sim\sqrt{2\pi}e^{-az}(az)^{az+b-1/2} \tag{48}$$

in Eq. (46) to find that $q^*(=D_f)\to \ln m/\ln n$. This coincides with the fractal dimension of the classic Cantor set. That is, in the limit $\beta\to\infty$ the standard deviation of the size of the pieces created tends to zero. This is easily verified by calculating the mean and standard deviations from $F\{x_i\}$. This particular finding implies that in this limit $F_n(x)$ behaves approximately as

$$F_n(x)=x^{\lambda}\int \delta(x_1-x_3)dx_1\int \delta(x_1-x_2)dx_2\cdots \times\int \delta\left(x_1-\left(x-\sum_{i=1}^{n-1}x_i\right)\right)dx_{n-1}. \tag{49}$$

Substituting this into Eq. (36), we obtain

$$\frac{\partial\psi(x,t)}{\partial t}=-\frac{x^{\lambda}}{n}\psi(x,t)+m(nx)^{\lambda}\psi(nx,t). \tag{50}$$

This describes a model that splits an interval into n equal pieces and keeps only m of them. Substituting the definition of the moment (6) into Eq. (50), we obtain

$$\frac{dM(q,t)}{dt}=\left[\frac{m}{n^{q+1}}-\frac{1}{n}\right]M(q+\lambda,t). \tag{51}$$

The solution of Eq. (51) is

$$M(q,t)\sim t^{-(q-D_f)/\lambda} \tag{52}$$

and immediately reveals that $D_f=\ln m/\ln n$, as in the classic Cantor set with kinetic exponent $1/\lambda$. In the limit $\beta+1\to 0$ we can analyze Eq. (46) to show that q^* (and D_f) tends to zero like $\gamma(m,n)(\beta+1)$, where $\gamma(m,n)=n(m-1)/(n-m)$. Consequently, we see that for all m and n there is a spectrum of fractal dimensions between $\beta\to -1$, when $D_f\to 0$, and $\beta\to\infty$, when $D_f\to\ln m/\ln n$. This is a very striking result. It implies that in the limit $\beta\to\infty$ particles are deposited only in the center of the empty space and produce strictly self-similar patterns.

TABLE II. Fractal dimension D_f of stochastic fractals for $m=2$ and $n=3$ for increasing β values.

β	D_f
$-\frac{1}{2}$	$\frac{1}{2}$
0	0.5616288
$\frac{1}{2}$	0.5841
1	0.5956
2	0.6073
∞	0.6309

VI. SUMMARY AND DISCUSSION

In this work we presented a number of interesting results relating to random and correlated sequential deposition of a mixture of particles of finite sizes. We found that the pattern created is statistically scale invariant. We also attempted to show the relationship between the fractal dimension and the degree of order in the resulting pattern created in the long-time limit. In Table I we presented some values of the fractal dimension as a function of m and n for model I (or for $\beta=0$ using model III) with $\alpha=1$. The corresponding fractal dimension for the Cantor set are given in parentheses. In Table II we give the fractal dimension for $m=2$ and $n=3$ for some different values of β. Table II and a more numerical survey confirm that the fractal dimension increases monotonically as β increases. Moreover, the fractal dimension appears to decrease as n increases for a given m and vice versa, provided $n-m>0$. In Table III we compare the fractal dimensions for different values of α for models I and II. This table and further details of the numerical survey confirm that for the same α value, model II creates a pattern that has a higher fractal dimension than that for model I. We further notice that the first row of Table II for fractal dimension with $m=2$ is exactly the same as in Table III with $\beta=0$ and $\alpha=1,2,3$. Similarly, the fractal dimension obtained from model III for $\beta=1$, $m=2$, and $n=(\alpha+5)/2$ would be the same as that obtained from model II for the corresponding $(\alpha+1)/2=1,2,\ldots$. This further shows that as α increases the length of the deposited particles on the average increases with respect to that of a corresponding average length for a lower α value, which appears to be consistent with our detailed survey that reveals that fractal dimension decreases monotonically as α increases, and as $\alpha\to\infty$ the fractal dimension $D_f\to 0$. These observations immediately confirm that the exponent α does not play any role in creat-

TABLE III. Fractal dimension D_f for $\beta=0,1$ for different values of α.

α	$\beta=0$	$\beta=1$
1	0.5616	0.5956
2	0.4348	0.5183
3	0.3723	0.466542
4	0.33405	0.429121
5	0.30784	0.400614
6	0.288505	0.37805
7	0.27351	0.35966

ing the ordered pattern since the fractal dimension does not reach a constant nonzero value as $\alpha \to \infty$.

In a recent Letter [21], Brilliantov *et al.* studied the random sequential adsorption of a mixture of particles with a continuous distribution of sizes determined by the power-law form [Eq. (3)]. They reported that the pattern created in the long-time becomes more and more ordered as α increases and in two dimensions it reaches the Apollonian packing in the limit $\alpha \to \infty$ when the depositing particles are a finite mixture of disks. It is important to notice that the one-dimensional analog of this model is the deposition of a mixture of rods. Evidently, one expects the fractal dimension to coincide with the classic Cantor set when the pattern becomes a perfectly ordered pattern. In Ref. [21] an exact expression for the fractal dimension is also given for the general d dimension. It is also clear that the Apollonian packing is a nonrandom or strictly self-similar fractal-like Sierpinsky gasket difference that lies only in the geometry of the depositing particles. The observation that the pattern becomes more and more ordered should be true for any dimension and for any geometry including $d=1$. That is, in order to support the result that in the limit $\alpha \to \infty$ the pattern becomes more and more ordered, the fractal dimension must reach a constant nonzero value in that limit. In particular, as $\alpha \to \infty$ the fractal dimension must reach the value ln2/ln3, which is the one-dimensional analog of both the Apollonian packing and Sierpinsky gasket. In one dimension we can solve the model exactly, which corresponds to our model I. The exact enumeration of the fractal dimension and detailed numerical survey reveals that $D_f \to 0$ as $\alpha \to \infty$ instead of reaching a constant nonzero value. Therefore, the analysis we give in this work contradicts the result reported in [21]. More recently, deposition of a finite mixture of rectangles in a two-dimensional substrate is studied in Ref. [22]. Although in this work particles are allowed to deposit on only one of the four corners, the model in Ref. [21] retains the generic features of deposition phenomena of a definite mixture and in particular is very close to the model we consider in this work. The work in Ref. [22,23] is the stochastic counterpart of the deterministic Sierpinsky carpet or Cantor gasket [1], which are strictly self-similar. In this deterministic case, the initiator is a square and the generator subdivides at each step into b^2 equal squares, p of which are removed according to a fixed rule. After an infinite number of iterations the resulting set can be seen as a generalization of the Cantor set to two dimensions that has the fractal dimension $D_f = \ln(b^2 - p)/\ln b$. However, in the case of its stochastic counterpart we have shown that the system does not reach simple scaling; instead the system shows multiscaling, a result that we believe to be true for the deposition of a finite mixture of particles of any geometry in more than one dimension. That is, the pattern created in the long-time limit has a global scaling exponent D_f and a local scaling exponent known as $f(q)$, where q is the Holder exponent. That is, the pattern can be divided into a subset that scales with different fractal dimension, a phenomena called multifractality[2,15]. However, in the case of a strictly self-similar pattern the system reaches a simple scaling behavior. The study further revealed that the global exponent or the fractal dimension of the random fractal is always lower than its corresponding strictly self-similar counterpart. Therefore, we conclude that the fractal dimension must increase with increasing order and reaches a maximum value when the pattern is in perfect order. In this work we show that the exponent α does not play any role in creating an ordered pattern. Instead, it implies that the length of the deposited particle at each time step increases on average as the α value increases.

In order to create an ordered pattern we reveal that one needs to choose $F(x,y|z) \sim (xy)^\beta$ and $p(z) \sim z^\beta$. This model for $\beta \neq 0$ describes that at each event an empty space is chosen randomly. However, once this decision has been made, particles are more likely to deposit at the center of the empty space than on either side of it. Of course, the tendency to deposit at the center increases as β increases. In fact, our analysis further supports that the fractal dimension increases with the degree of increasing order and reaches its maximum value (a nonzero constant) in the perfectly ordered pattern, as it does in the classic Cantor set or in the Seirpinsky gasket. Krapivsky and Ben-Naim reported in [12] that the dimension of the random fractal is always smaller than its deterministic counterpart. If we look at the situation for the stochastic Cantor set, we find that in order to get the deterministic classic set ($\ln m/\ln n$) α alone does not play any role in creating any ordered pattern. Instead, one has to choose $F(x,y|z)$ to be the same power-law form as for the parking distribution $p(z)$, e.g., model III. In this case only the β value determines the degree of tendency to place the particles in the center of the empty space and in the limit $p \to \infty$ particles are always placed exactly at the center of the empty space. We can quantify the increasing regularity of the resulting pattern created in the long-time limit by introducing the concept of entropy production that characterizes the degree of order as

$$S = -\sum_{C_k} p(C_k) \ln p(C_k). \tag{53}$$

Since in the $\beta \to \infty$ limit there is only one definite configuration, we have $p(C_k)=1$, which contributes to the entropy. In fact, we can define

$$p(C_k) = \frac{D_f}{\ln m/\ln n}, \tag{54}$$

where each D_f corresponds to one definite configuration. We find that S increases as β decreases towards zero. That is, as β decreases the number of possible configurations increases. In this work, we generalize the conventional random sequential absorption where the position of a particle to be deposited in the empty space is chosen randomly and the degree of randomness is determined by the position-dependent deposition rate. That is, the position where the particle is to be placed is chosen by the size of the empty space being destroyed and by the size of the empty spaces created on either side. As a prospect of future work one can choose $p(z) = \delta(x-1)$ and $F(x,y|z)$ to be position dependent. Studying random sequential absorption with these choices, the system obviously will reach a jamming limit, but how it varies with the degree of order can be of greater physical interest.

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